Simulations and experiments of short intense envelope solitons of surface water waves

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The problem of existence of stable nonlinear groups of gravity waves in deep water is considered by means of laboratory and numerical simulations with the focus on strongly nonlinear waves. Wave groups with steepness up to $A_{cr}\omega_m^2/g \approx 0.30$ are reproduced in laboratory experiments ($A_{cr}$ is the wave crest amplitude, $\omega_m$ is the mean angular frequency, and $g$ is the gravity acceleration). We show that the groups remain stable and exhibit neither noticeable radiation nor structural transformation for more than 60 wavelengths or about 15-30 group lengths. These solitary wave patterns differ from the conventional envelope solitons, as only a few individual waves are contained in the group. Very good agreement is obtained between the laboratory results and numerical simulations of the potential Euler equations. The envelope soliton solution of the nonlinear Schrödinger equation is shown to be a reasonable first approximation for specifying the wave-maker driving signal. The short intense envelope solitons possess vertical asymmetry similar to regular Stokes waves with the same frequency and crest amplitude. Nonlinearity is found to have remarkably stronger effect on the speed of envelope solitons in comparison to the nonlinear correction to the Stokes wave velocity. © 2013 AIP Publishing LLC.

I. INTRODUCTION

The paper is aimed at studying experimentally and numerically the conventional notion of envelope solitons in the context of surface water gravity wave groups. It is well known that the envelope soliton is an exact solution of the integrable nonlinear Schrödinger equation (NLS), which is the basic model that describes weakly nonlinear water waves propagating in one direction.1–3 In the infinite line NLS equation framework, envelope solitons compose the major part of the wave field for the Cauchy problem in the limit of very long time.4 The possibility of finding solution of the NLS equation by means of the Inverse Scattering Transform makes envelope solitons extremely attractive objects which are useful for the comprehension of nonlinear wave dynamics, see, e.g., Refs. 5 and 6.

Weakly nonlinear solitary groups were tested in the past in laboratory conditions, see Refs. 7–9; they were also examined in numerical simulations within different frameworks.10–12 The general conclusion to be drawn from such studies is that when transverse direction effects are neglected, weakly nonlinear wave groups do exhibit some structural stability, i.e., they behave as prescribed by the weakly nonlinear models and propagate without noticeable distortion and survive after collisions.

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At the same time, relatively steep solitary groups, \(k_0A_0 > \sim 0.2\) (where \(k_0\) is the carrier wavenumber and \(A_0\) is the soliton amplitude) show dispersive properties.

Despite the preceding knowledge, very short stable wave groups that contain a very small number of steep waves (up to the breaking limit) have been found in recent fully nonlinear simulations of the potential Euler equations by Dyachenko and Zakharov.\(^{13}\) In Ref. 14 the short intense solitary wave groups were associated with the strongly nonlinear limit of envelope soliton solutions of the NLS equation. It was found that this analytic solution provides a rather accurate initial condition for simulating strongly nonlinear coherent wave groups, which behave as envelope solitons, even when the wave steepness approaches the value of about \(k_0A_0 \sim 0.3\), i.e., well beyond the formal validity of the NLS equation. Third order bound waves determined within the generalized NLS theory (see, e.g., Refs. 15–17) were employed to improve description of water waves. For steeper initial conditions the wave group underwent to breaking at the early stage of evolution. Comparing the fully and weakly nonlinear simulations, it was found in Ref. 14 that the dynamics of these strongly nonlinear solitary wave groups was rather well captured by the modified NLS equation (the generalized Dysthe model\(^{15,17,18}\)) up to the steepness about \(k_0A_0 \sim 0.2\).

Summarizing, the question of existence of steep and short solitary wave trains has not been clarified yet. On one side, recent fully nonlinear simulations of unidirectional waves predict the existence of very steep and short solitary structures with the maximum amplitude limited by the wave breaking effect. The numerical results allow to consider the NLS soliton solution as a reasonable first approximation for these groups even in the case of rather steep waves \((k_0A_0 \sim <0.2)\). On the other hand, no laboratory measurements of strongly nonlinear envelope solitons have been performed, and it is not clear whether those structures can exist in nature.

The laboratory experiments reported in this paper pursue two main objectives:

- reproduce the “limiting” envelope solitons, i.e., very short groups of very intense waves (as intense as possible, with steepness exceeding the value of \(k_0A_0 = 0.2\)), which propagate with persistence;
- test the NLS envelope soliton solution as the boundary condition for generating nonlinear solitary wave groups with the focus on the case of large wave steepness.

The steep solitary wave groups, which we consider in the present work, contain very small number of individual waves. Strongly nonlinear numerical simulations are employed in the present study with the primary intention to compare the laboratory results with the theory.

The paper is organized as follows. Section II reminds briefly some features of the weakly nonlinear theory described by the NLS equation, in particular, the envelope soliton solution is introduced. Stationary nonlinear wave groups, which are obtained in numerical simulations of potential Euler equations, are analyzed and discussed in Sec. III. The experimental setup and results of laboratory measurements of intense wave groups are given in Sec. IV. They are also compared with the results of numerical simulations given in Sec. III. Section V contains the summary of findings and the discussion.

**II. WEAKLY NONLINEAR THEORY FOR ENVELOPE SOLITONS**

The NLS equation,

\[
i \left( \frac{\partial A}{\partial t} + C_{gr} \frac{\partial A}{\partial x} \right) + \omega_0 \frac{\partial^2 A}{\partial x^2} + \frac{\omega_0 k_0^2}{2} |A|^2 A = 0,
\]

(1)

represents the leading-order theory for the description of unidirectional gravity water waves under the assumptions of small nonlinearity, \(k_0|A| \ll 1\) (where \(A\) is the amplitude of water displacement), and narrow spectral bandwidth, \(\Delta k/k_0 \ll 1\). Here, the carrier wavenumber, \(k_0\), and characteristic wavenumber spectrum width, \(\Delta k\), are introduced. The group velocity of linear waves, \(C_{gr} \equiv \omega_0/(2k_0)\), is provided by the deep-water dispersion relation

\[
\omega_0 = \sqrt{g k_0}.
\]

(2)
In order to compare solutions of the weakly nonlinear theory with the results of laboratory experiments, it is useful to re-write the NLS equation (1) in the form, which describes the evolution of wave groups in space,

\[ i \left( \frac{\partial A}{\partial x} + \frac{1}{C_{gr}} \frac{\partial A}{\partial t} \right) + \frac{\omega_0}{8k_0^3 C_{gr}^3} \frac{\partial^2 A}{\partial t^2} + \frac{\omega_0 k_0^2}{2C_{gr}} |A|^2 A = 0. \]  

(3)

From a mathematical point of view, Eq. (3) has the same asymptotic validity as Eq. (1).

The complex-valued function of coordinate and time, \( A(x, t) \), determines both, the surface elevation, \( \eta(x, t) \), and the velocity potential \( \varphi(x, z, t) \); in particular, the surface elevation is given by

\[ \eta = \text{Re}(A \exp( i \omega_0 t - i k_0 x)). \]  

(4)

The fields \( \eta \) and \( \varphi \) may be computed with higher accuracy when a high order NLS theory is applied. The third-order asymptotic theory determines the surface elevation according to the following relations:

\[ \eta(x, t_0) = \frac{1}{4} \hat{H} \left\{ \frac{\partial |A|^2}{\partial x} \right\} + \text{Re}(A \exp(i \omega_0 t_0 - i k_0 x)) + \frac{k_0}{2} \text{Re} \left( A^2 \exp(2i \omega_0 t_0 - 2i k_0 x) \right) - \frac{1}{2} \text{Im} \left( A \frac{\partial A}{\partial x} \exp(2i \omega_0 t_0 - 2i k_0 x) \right) + \frac{3k_0^2}{8} \text{Re} \left( A^3 \exp(3i \omega_0 t_0 - 3i k_0 x) \right), \]  

(5)

\[ \eta(x_0, t) = -\frac{k_0}{2\omega_0} \hat{H} \left\{ \frac{\partial |A|^2}{\partial t} \right\} + \text{Re}(A \exp(i \omega_0 t - i k_0 x_0)) + \frac{k_0}{2} \text{Re} \left( A^2 \exp(2i \omega_0 t - 2i k_0 x_0) \right) + \frac{k_0}{\omega_0} \text{Im} \left( A \frac{\partial A}{\partial t} \exp(2i \omega_0 t - 2i k_0 x_0) \right) + \frac{3k_0^2}{8} \text{Re} \left( A^3 \exp(3i \omega_0 t - 3i k_0 x_0) \right). \]  

(6)

Formula (5) is used for given function of space, \( A(x, t_0) \), and formula (6) is used for function of time, \( A(x_0, t) \). Here, terms of order \( O((k_0 A)^3) \) and \( O((\Delta k/k_0)^3) \) are included, and \( \hat{H} \) is the Hilbert transform of corresponding coordinate, \( x \) or \( t \), which takes into account the long-scale induced flow (see details in Refs. 16 and 17).

The classical NLS equation is able to describe nonlinear wave dynamics accurately for a rather short distance of wave propagation (of the order of \( A^{-2} k_0^{-3} \), according to the laboratory investigation in Ref. 19, i.e., a characteristic “nonlinear distance”). The Dysthe equation, which is the next-order generalization of the NLS model, provides much better description, see, e.g., Refs. 19 and 20. Other generalizations of the high order NLS theory are possible; often they represent a reasonable compromise between efficiency and simplicity.

The NLS equation is integrable by means of the Inverse Scattering Technique and possesses the envelope soliton solution in the form of a sequence of waves that constitute a stable wave group (see, e.g., Refs. 5 and 6). For equations in forms (1) and (3) the envelope soliton solutions are given by expressions, respectively,

\[ A(x, t) = A_0 \exp \left[ i \frac{s_0^2}{4} \omega_0 t \right] \frac{\cosh \left[ \sqrt{2s_0 k_0} (x - C_{gr} t) \right]}{\cosh \left[ \sqrt{2s_0 k_0} (x - C_{gr} t) \right]}, \]  

(7)

\[ A(x, t) = A_0 \exp \left[ i \frac{s_0^2}{2} k_0 x \right] \frac{\cosh \left[ \sqrt{2s_0 k_0} (x - C_{gr} t) \right]}{\cosh \left[ \sqrt{2s_0 k_0} (x - C_{gr} t) \right]}, \]  

(8)

where \( A_0 \) is the soliton amplitude; the characteristic wave steepness of the soliton may be specified by \( s_0 \equiv k_0 A_0 \). Envelope solitons (7) and (8) propagates with the group velocity of linear waves, \( C_{gr} = \omega_0 / (2k_0) \).
Within the framework of the integrable NLS equation envelope solitons interact elastically between each other and with other quasi-linear waves. In contrast to transient wave groups, the envelope soliton is constituted by interacting coherent wave harmonics that prevents dispersion of the group. The Fourier spectrum of group (7) may be obtained straightforwardly,

\[ \hat{A}(k, t) \equiv \int_{-\infty}^{\infty} A(x, t) e^{ikx} dx = B \exp(i\theta), \]

\[ B(k) = \frac{\pi A_0}{\sqrt{2}s_0k_0 \cosh \frac{\pi k}{2\sqrt{2}s_0k_0}}, \]

\[ \theta(t) = \frac{s_0^2}{4} \omega_0 t - kC_{gr} t. \]

Hence, all Fourier modes have the same phases. It is remarkable that the Fourier amplitudes \(B(k)\) do not evolve in time, thus, the NLS envelope soliton corresponds to a stationary solution in the Fourier space with co-phased harmonics.

Because Eq. (3) differs from Eq. (1) only in coefficients, solution (9) may by straightforwardly re-written for the boundary problem described by Eq. (3).

### III. NUMERICAL SIMULATIONS

The numerical simulation of the primitive water equation is used in this section with the purposes of verifying the theory and of obtaining the information that is difficult to be measured directly in laboratory experiments. The algorithm employed for solving the potential Euler equations for infinitively deep water in a periodic spatial domain is the High Order Spectral Method (HOSM) described by West et al.\(^{11}\) The nonlinear parameter, which controls the number of terms in the expansion of the velocity potential near the undisturbed level, is set equal to six, \(M = 6\), which corresponds to consider up to 7-wave interactions. The HOSM is a rigorously derived,\(^{11}\) well-documented, and widely used method. It allows accurate description of waves with steepness at least of 0.3. Some comparison was made between the HOSM and the simulations in conformal variables (long travelling distance, without the padding mask). Some insignificant deviations were observed for steepest cases.

The envelope soliton (7) provides the initial condition for the numerical simulations. Namely, the surface elevation, \(\eta(x)\) (obtained with the help of reconstruction formula (5)) and the surface velocity potential, \(\phi(x) = \psi(x, z = \eta)\) (the reconstruction formula is not given in the text, but may be found in Refs. 16 and 17), are used to start the simulation at \(t = 0\). The value of the carrier wavenumber \(k_0 = 1\) rad/m is used in all numerical simulations; the gravity acceleration is specified as \(g = 9.81\) ms\(^{-2}\).

Intense solitary groups were simulated in Ref. 14 within the framework of the HOSM and also using a fully nonlinear code of the Euler equations in conformal variables. Both the approaches showed quite similar wave dynamics, i.e., at the early stage of the evolution the initial wave group underwent some relatively weak radiation in both directions (quasi-linear waves which propagate with group velocities both larger and smaller than that of the soliton); later on, the radiated component was spread throughout the computational domain. If the initial wave group was steeper than about \(k_0 A_0 \approx 0.3\), then the wave breaking was observed shortly after the start of simulations. If the wave survived after the initial stage, then it was observed that the solitary wave group stabilizes and propagates over the background of the radiated component for a long time without any evidence of energy leakage. As the radiated component observed in Ref. 14 was reasonably small, the NLS solution is considered hereby as the first approximation for steep solitary wave groups. In contrast to Ref. 14, in the present study a moving padding frame is introduced, which damps all waves aside the area close to the desired wave group. A qualitatively similar approach was used by Dyachenko and Zakharov.\(^{13}\)

In our simulations the evolution of the wave group is first simulated for a couple of hundred of wave periods with the purpose of letting it adjust and to obtain the shape of the “true” stationary wave group. During this period some wave energy which is radiated away from the group is adsorbed
due to the padding mask, and hence the total wave energy decays with time, see solid line in Fig. 1. In this figure the temporal evolution of the potential (dashed-dotted line), kinetic (broken line), and total (solid line) energies is shown normalized by the corresponding initial values. The figure reveals that the initial condition does not provide the proper balance between potential and kinetic energies, as the corresponding curves immediately separate from each other. The total wave energy loss in the case shown in Fig. 1 is about 3%, which is about twice smaller than the difference between the kinetic and potential energies. Figure 1 is related to a fairly nonlinear wave group, \( k_0 A_0 = 0.3 \); other properties of the observed stationary wave groups are given in Table I, see experiment No. 9. The energy dissipation and the difference between kinetic and potential energies are smaller if less intense waves are simulated; the latter confirms that nonlinear effects are responsible for the observed difference between the potential and kinetic parts of the energy.

After the stage of adjustment (about 200 wave periods), we start to store the simulated data with short time intervals (for the following about 50 wave periods) with the purpose of obtaining the high-resolution picture of the wave dynamics. For this stage the total energy is shown by a thicker line in Fig. 1. Some small fluctuations of potential and kinetic energies within the scale of a wave period may be seen in the figure. The values of the energies are almost constant for \( t > 200 T_p \), which implies a quasi-stationary state. In all cases solitary wave groups were formed in the course of the evolution. The data accumulated during the stage from 200 to 250 wave periods is used for the investigation of properties of the stationary wave groups.

Rather steep wave groups are simulated in the paper, with the initial condition in the form of a NLS soliton (7) and characteristic steepness in the range \( 0.15 \leq k_0 A_0 \leq 0.35 \), see Table I. Instability

![FIG. 1.](image)

FIG. 1. Evolution of the potential, kinetic, and total energies (see the legend) in numerical simulations; variations with respect to the corresponding initial values. Experiment No. 9 from Table I is shown (\( k_0 A_0 = 0.3 \)).

### Table I. Characteristics of stationary wave groups observed in numerical simulations.

<table>
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<tr>
<th>Numerical experiment No.</th>
<th>( k_0 A_0 )</th>
<th>( k_0 A_0^{(3)} )</th>
<th>( k_0 A_0^{(3)} )</th>
<th>( A_0 )</th>
<th>( A_0 )</th>
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<th>( \omega )</th>
<th>( V )</th>
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<td>0.15</td>
<td>0.16</td>
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<td>0.99</td>
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<td>0.18</td>
<td>0.19</td>
<td>0.16</td>
<td>0.17</td>
<td>0.97</td>
<td>0.98</td>
<td>3.07 3.17 1.62</td>
</tr>
<tr>
<td>4</td>
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<td>0.25</td>
<td>0.20</td>
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<td>0.20</td>
<td>0.97</td>
<td>0.98</td>
<td>3.13 3.18 1.63</td>
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<td>0.21</td>
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<td>0.28</td>
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<td>0.26</td>
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<td>3.07 3.22 1.67</td>
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<tr>
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<td>0.27</td>
<td>0.33</td>
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<td>0.29</td>
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<td>0.97</td>
<td>3.07 3.25 1.69</td>
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<td>0.96</td>
<td>3.07 3.31 1.72</td>
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of numerical simulations, associated with the wave breaking phenomenon, is observed in the case $k_0A_0 = 0.35$. The steepest reliable stable case in the simulations corresponds to $k_0A_0 = 0.32$. Hence, some natural maximum amplitude limit of the solitary wave groups is found in the interval of initial parameters $0.32 < k_0A_0 < 0.35$.

The evolution of different characteristics of wave intensity is displayed in Figs. 2(a) and 2(b) for cases $k_0A_0 = 0.2$ and $k_0A_0 = 0.3$, respectively. Four features of the wave field intensity are considered: (i) wave crest steepness, $k_m A_{cr}$, (ii) wave trough steepness, $k_m A_{tr}$, (iii) steepness specified on the basis of the dimensionless half-height, $k_m H_x/2$, and (iv) the maximum local slope of the surface displacement, $\partial \eta / \partial x$. Quantities $A_{cr}$ and $A_{tr}$ are the zero-crossing crest and trough amplitudes, respectively. Hereafter, $H_c$ and $H_t$ correspond to the wave heights determined on the basis of spatial or time series, respectively. Each of them corresponds to the maximum value between the up-crossing and down-crossing heights. The frequency and wavenumber Fourier spectra of the stationary wave groups are used to obtain the peak ($\omega_p$, $k_p$) and mean ($\omega_m$, $k_m$) values of the angular frequency and the wavenumber, respectively. The mean values $\omega_m$ and $k_m$ are obtained as the first moments of the corresponding power spectrum, see Table I. These quantities differ from the initially specified carrier wavenumber, $k_0$, and the corresponding frequency of linear waves, specified by relation (2). In the simulations values $\omega_p$ are found to be slightly smaller than the linear frequency of initial waves, $\omega_0$. The wavenumber $k_p$ coincides with $k_0$ at $t = 0$, but later on the actual peak wavenumber, $k_p$, becomes slightly less than $k_0$. The mean frequency and wavenumber are slightly larger than the peak values, see Table I. The corresponding wave periods are defined by $T_p = 2\pi / \omega_p$ and $T_m = 2\pi / \omega_m$; the peak wavelength is $\lambda_p = 2\pi / k_p$.

Amplitude characteristics of the steady nonlinear groups observed in numerical simulations are shown in Fig. 2 as functions of time. It may be pointed out that these characteristics reach local extremes at close but different times; the wave height oscillates twice more frequently than the other observables. During the displayed time interval, some weak variability of the wave group can be appreciated in Fig. 2(b).
The parameter $A_0$, which specifies the initial condition in the form of a soliton (7), cannot be measured directly due to the existence of phase-locked bound waves. Let us estimate the amplitude of observed waves with better accuracy and to estimate the drop of wave intensity due to energy loss. For simplicity, in below estimations, we take into account bound waves for uniform Stokes waves up to the cubic nonlinear correction (see, e.g., Refs. 16 and 17). The bound wave components alter the Stokes wave shape as follows:

$$kA^{(3)}_{c} = kA + \frac{1}{2} k^2 A^2 + \frac{3}{8} k^3 A^3,$$

$$kA^{(3)}_{r} = kA - \frac{1}{2} k^2 A^2 + \frac{3}{8} k^3 A^3,$$

where $k$ is the characteristic wavenumber.

For the case $k_0A_0 = 0.3$ (see experiment No. 9 from Table I) the initial wavenumber is $k_0 = 1 \text{ rad/m}$, and the wavenumber of the stationary wave group in the simulations is $k_p \approx 0.92 \text{ rad/m}$, and therefore $k_pA_0 \approx 0.276$. The steepness of the maximum wave in the initial group is estimated with use of Eq. (10) as $k_0A^{(3)}_{cr} \approx 0.355$ and $k_0A^{(3)}_{tr} \approx 0.265$, and for the actual peak wavenumber of the solitary group the values are $k_pA^{(3)}_{cr} \approx 0.323$ and $k_pA^{(3)}_{tr} \approx 0.246$. Hence, in Fig. 2(b) we show a significant decay of the wave steepness in comparison to the initial condition.

A similar estimation for the case $k_0A_0 = 0.2$ (experiment No. 3 from Table I) gives values $k_0A^{(3)}_{cr} \approx 0.223$ and $k_0A^{(3)}_{tr} \approx 0.183$ versus $k_pA^{(3)}_{cr} \approx 0.215$ and $k_pA^{(3)}_{tr} \approx 0.178$. The values for other numerical experiments are reported in Table I.

Due to the strong asymmetry of waves, it is not straightforward to apply the concept of wave envelope using, for example, the Hilbert transform. Therefore, in the present study wave envelopes are obtained by assembling all the surface elevations during the last 50 wave periods of simulations and all coordinates, $x$, plotted in the reference frame, which moves with velocity of the group, $V$. This velocity is obtained as the speed of propagation of the wave group “center of mass”:

$$x_s(t) = \frac{\int x\eta^2(x,t)dx}{\int \eta^2(x,t)dx},$$

where the integration is performed within the spatial domain of the computation, taking into account the periodic boundary conditions. The sequence of positions $x_s(t)$ is interpolated by the linear relation $x_s = x_0 + Vt$ that furnishes the value of the wave group velocity, $V$. The obtained solitary group velocities (see Table I) are discussed in Sec. IV and are compared with results of the laboratory measurements (see Fig. 9).

The observed travelling groups are very short wave packets of strongly nonlinear waves, and hence, as mentioned before, the concepts of wave modulation and wave envelope cannot be applied to the case in a straightforward manner. The maxima and minima of the surface elevations for a given coordinate/time in the co-moving frame provide the upper and lower enveloping curves correspondingly. These envelope shapes for stationary wave groups as functions of time are shown in Fig. 3(a). The stationary wave groups shown in Fig. 3(a) result from initial conditions in the form of NLS solitons with parameters $k_0A_0$ from 0.15 to 0.32, see Table I. The envelopes become higher and shorter as the initial steepness is larger.

Figure 3(b) estimates the vertical asymmetry of the envelopes (stars). Note that here values $A_{cr}$ and $A_{tr}$ are amplitudes of the upper envelope curve in Fig. 3(a) and of the corresponding lower one. In other words, values $A_{cr}$ and $A_{tr}$ are the maximum crest height and the deepest trough of the individual waves which may be observed in the envelope in the course of its evolution. The horizontal axis in Fig. 3(b) represents the dimensionless crest amplitudes scaled with the peak frequency. These characteristics of wave shapes are collected in Table I, scaled with the mean frequency. The quantity $\max(H(t))$ estimates the maximum dimensionless wave height in time series of wave groups.

The vertical asymmetry of uniform Stokes waves, $A_{cr}/A_{tr}$, is given by the dashed line in Fig. 3(b) for reference; for this line the horizontal axis corresponds to the wave crest amplitude, scaled in a similar fashion. $A_{cr} \omega_p^2 / g$, where $\omega_p$ is the frequency of the Stokes wave with the numerically determined strongly nonlinear correction taken into account. When in the horizontal axis in Fig. 3(b) the scale of $\omega_m$ is used instead of $\omega_p$, the symbols are placed a little bit lower than the dashed line. Estimations of the mean frequency and the mean wavenumber are more stable.
thus are assumed to be more reliable than the ones of the peak values. These mean quantities are used in Sec. IV, where laboratory measurements of propagating intense wave groups are discussed.

Figure 3(b) shows that the vertical asymmetry of the solitary wave groups agrees very well with the asymmetry of Stokes waves with the same crest amplitudes and the same peak frequencies. In accordance with this statement, the upper envelope, $A_{cr}$, will be used as a measure of wave group intensity, which is consistent with the Stokes wave crest amplitude, $A_{cr}$.

The stationary wave group which is generated in the case $k_0A_0 = 0.30$ is displayed in Fig. 4 in more detail. Figures 4(a) and 4(b) show the wave envelope (dashed line) and the surface elevation (solid lines) when the maximum wave crest (thick solid line) and the deepest wave trough (thin solid line) occur, as a function of coordinate and time, respectively. The different number of waves in the envelope in Figs. 4(a) (space series) and 4(b) (time series) is notable and it is associated with the difference between the phase and group velocities. The wave group in Fig. 4(a) is steep, and consists of only about three waves, while the wave sequence in Fig. 4(b) represents a visually smoother modulation of wave train.

The wavenumber and frequency spectra of corresponding surface elevations are shown in Figs. 4(c) and 4(d), respectively. The spectra are computed for two stages of the wave group evolution: when the crest is highest (solid lines) and when the trough is deepest (dashed lines). While the NLS approximate solution possesses a stationary spectrum (see Sec. II and formula (9)), the evolution of spectrum is evident in Figs. 4(c) and 4(d), and is most notable in the case of wavenumber spectrum. At the moment of the maximum wave crest the comb-shaped wavenumber spectrum becomes single-peaked. In the long-wave range the picture of spectrum evolution is inverse: the wavenumber spectrum has larger values at the moment of deepest trough. In the frequency spectrum the second harmonic remains detached, and variation in the low-frequency range is negligible.

The spectrum evolution within a wave period becomes less visible for smaller waves. In the case $k_0A_0 = 0.20$ the evolution cannot be seen by eye in the frequency spectrum, but may be noticed in the wavenumber spectrum plot. The frequency spectrum shows that its evolution is most likely due to the overlap between the free and bound wave components, which occurs differently at different phases of the wave group evolution.
FIG. 4. The stationary wave group generated from the initial condition characterized by $k_0A_0 = 0.30$ (experiment No. 9 from Table I). Surface elevations (solid lines) and wave envelopes (dashed lines) are given in panels (a) and (b) as functions of coordinate and time, respectively. The wavenumber spectrum and frequency spectrum at the moments of maximum wave crest (solid lines) and the deepest trough (dashed lines) are shown in panels (c) and (d).

IV. LABORATORY TESTS

The laboratory tests of steep and short solitary-like groups of waves were performed in the seakeeping basin of the Technical University of Berlin. The basin is 110 m long, with a measuring range of 90 m. The width is 8 m and the water depth is 1 m. At one end, a fully computer controlled electrically driven wave generator is installed which can be utilized in piston as well as flap type mode. On the opposite side, a wave damping slope is installed to suppress disturbing wave reflections. For this test campaign the wave generator is driven in flap type mode and the center of rotation is on the bottom of the basin as the wave board covers the full water depth.

The boundary condition for the wave maker, i.e., the wave sequence at the wave board, was obtained according to the following three different approaches:

1. in the form of the NLS envelope soliton (Eq. (8)) for the free wave component; the surface elevation is given by (4), $\eta = \text{Re}(A(t) \exp(i\omega_0 t))$ (Method 1);
2. with the use of time series of the surface elevation obtained through Euler simulations of stationary wave groups, as it is described in Sec. III (Method 2);
3. in form of the NLS envelope soliton (8), when the surface elevation \( \eta(t) \) is reconstructed with the help of formulas (6) which takes into account three asymptotic orders of the generalized NLS theory\(^{17} \) (Method 3).

In different runs wave groups in two different phases (highest crest or deepest trough) were reproduced at the wavemaker.

Multiplication of the calculated wave sequence with the hydrodynamic as well as electric transfer function of the wave generator in frequency domain and subsequent Inverse Fast Fourier Transformation result in the control signal for the wave generator. The hydrodynamic transfer function is modelled using the Biesel function,\(^{21} \) relating the wave board stroke to the wave amplitude at the position of the wave maker. The obtained control signal is afterwards checked against the wave generator limitations – maximum wave board velocity and acceleration – to ensure a smooth operation as well as non-breaking waves at the wave board. The exact transfer of calculated boundary conditions to the seakeeping basin is a delicate procedure, in particular, for this test campaign, as deviations at the beginning will influence the complete propagation of the wave group along the basin. One point thereby is the application of linear transfer functions – for steeper waves the control signal may become more inexact in comparison to less steeper waves. But the main point is the fact that the wave generator provides a velocity profile at the wave board which differs significantly from the “natural” one under the same surface elevation depending on the geometry and type of the wave board. In this study the flap type mode of the wave board generates a velocity profile which decays linearly from top to bottom. The deviation between the flap motion and the exact particle motion under the surface elevation causes a first order disturbance at the wave board, which decays during propagation of the waves and is theoretically zero after a distance of only one wavelength from the wave maker.\(^{22} \) The influence of second order disturbances is significantly reduced due to the fact that the flap type mode is chosen and all generated wavelengths are almost in the deep water domain (all waves are deep-water waves regarding the criterion \( h/\lambda > 0.5 \), where \( h \) is the water depth. The longest carrier wavelength is about 1.8 m).\(^{21} \) So the waves need some time and space to become fully “physical” which maybe also influence the accuracy of the reproduced shape of the waves.

Figure 5 presents the general overview on the experimental setup. Altogether, ten wave gauges were installed: single devices at fetches 10, 30, 60, and 85 m, and an array of 6 closely situated gauges at the distance 45 m. The gauges are numerated hereafter in the order as they appear from the wave maker to the end of the tank. Special attention was paid on the position of wave gauge 1. As mentioned above, the selection of the first position follows two complementary requirements: on one hand one is interested in being as close as possible to the wave board for comparison between target and reproduced wave; on the other hand the wave needs some time to fully evolve and thus the wave gauge should be placed several wavelengths away from the wave board. For this study the first wave gauge is placed 10 m in front of the wave maker which means that the wave gauge is at least 5 wavelengths away from the wave maker.

The test program comprises the variation of the initial wave steepness \( (k_0A_0 = 0.15, 0.2, 0.25, 0.3, 0.35) \) and the carrier wave frequency \( (\omega_0 = 5.92, 6.82, 6.86, 7.52 \text{ rad/s}, \) and corresponding \( k_0h = 3.57, 4.74, 4.80, 5.76) \). The carrier frequencies are chosen in such a way that the relevant frequency bandwidth of the wave sequence spectrum is within the wave generator limitations.

![FIG. 5. Test setup – side view on the seakeeping basin with the wave generator on the left, the damping slope on the right, as well as the positions of the ten wave gauges installed for this test campaign.](image-url)
The lower reproducible frequency is 0.5 rad/s, and the highest frequency varies from 10 to 30 rad/s. Totally, 43 runs with different initial conditions were measured. The focus was made on generation of wave groups which would not exhibit significant structural variation along the tank, and on excitation of maximum steep steady wave groups. Only conditions without evidence of wave breaking were permitted. Parameters for the best experiments, when quasi-stationary wave groups were observed, and for one case of unstable wave packet, are given in Table II.

Many of the performed experimental runs exhibit significant destruction of the wave group as it propagates along the tank. Such an example is given in Fig. 6(a): ten time series of the surface elevation registered by gauges from 1 to 10 are shown. A radiated wave train behind the main wave group is well seen in Fig. 6(a) at gauges 1 and 2 (right side from the main group). Later on, a rather intense wave train outruns the main wave group at gauges 2-10. The initial condition in this case is generated according to Method 3.

Figure 6(b) shows measurements of a stable wave group which is observed in experiment No. 30.16. The initial signal is the time series taken from the numerical simulation of a stationary wave group (Method 2). Both experiments shown in Fig. 6 correspond to the same initial wave group steepness, $k_0A_0 = 0.3$.

For each experimental run, the velocity of the wave group, $V$, was obtained tracking its path, similar to the approach applied previously to the numerical experiments. Then, all the records from ten gauges were plotted moving with velocity $V$ reference system. The variability of the envelope in the course of the wave group propagation was estimated visually, and the cases characterized by quasi-permanent wave group shapes were picked out as the “best” ones (they are listed in Table II) and studied further. The surface elevation profiles for the best experiments are shown by thin solid lines in Figs. 7(a)–7(c) in co-moving coordinates. One of the time series, at gauge 3, is given by a thicker line.

For the case of initially relatively small-amplitude waves, $k_0A_0 = 0.20$, experiment Nos. 29.14 and 30.07 are found to be the best (Fig. 7(a)); for $k_0A_0 = 0.3$ – experiment Nos. 30.13 and 30.16 (Fig. 7(b), time series of the latter experiment are shown in Fig. 6(b)); for $k_0A_0 = 0.35$ – experiment No. 30.37 (Fig. 7(c)). Parameters of initial conditions for the experiments are given in Table II.

Envelopes, dashed curves in Figs. 7(a)–7(c), are the results of strongly nonlinear numerical simulations described in Sec. III. Note that the best agreement between the envelopes and the displacement is obtained when the steepness of the initial condition, $k_0A_0$, is different for the cases of laboratory experiments and the numerical simulations. The experimental cases shown in Fig. 6(a) ($k_0A_0 = 0.20$), Fig. 6(b) ($k_0A_0 = 0.30$), and Fig. 6(c) ($k_0A_0 = 0.35$) are compared to the numerical simulations of groups $k_0A_0 = 0.15$, $k_0A_0 = 0.23$, and $k_0A_0 = 0.29$, respectively. Thus, for the same initial conditions, stationary wave groups seem to have eventually smaller steepness in laboratory experiments in comparison to the numerical simulations. The wave steepness obtained in the physical basin is smaller most likely due to the above mentioned peculiarities of the boundary conditions of the wave generator or due to the wave dissipation near the wavemaker. For the three considered cases the steepnesses of stationary wave groups are $A_{cr} \omega^2/mg \approx 0.150, 0.235, 0.301$, respectively, see
FIG. 6. Time series of the surface elevation at different distances, measured in the laboratory tank: an unstable wave group (a) (experiment No. 30.29) and stationary wave group (b) (experiment No. 30.16). Both the cases correspond to \( k_0 A_0 = 0.3 \), but to different carrier wave frequencies and different methods of signal generation.

horizontal dotted lines in Fig. 7. The quantity \( \omega_m^2/g \) gives the estimation of the mean wavenumber, which is directly calculated on the basis of the measured time series.

A very good agreement between the sequences of wave elevation profiles from the laboratory experiments and the envelope shape obtained via the numerical simulations (using the adjusted wave steepness of the initial condition) may be pointed out. Figure 7(c) shows the most extreme non-breaking stationary wave group which was observed in the laboratory conditions. It fits well the extreme stationary wave group found in the numerical simulations.

The frequency spectrum for laboratory experiment No. 30.16 (\( k_0 A_0 = 0.30 \)) is shown in Fig. 8 in linear and semi-logarithmic scales; ten records from the gauges are given in one plot. Generally, two wave harmonics may be clearly distinguished in the spectra of stable wave groups, and the third harmonic is somewhat less evident, see Fig. 8(b). The spectrum in semi-logarithmic scales looks rather stationary, while in the linear scales some variations may be observed. Such variability seems to be much less pronounced in experiments with less steep wave groups (Nos. 29.14 and 30.07) in comparison with other steeper cases listed in Table II. Numerical simulations do not report such strong variability of the Fourier spectrum, see Figs. 4(c) and 4(d).

Velocities of the stationary wave groups observed in laboratory experiments are given by circles in Fig. 9 for the best selected cases of stable wave groups. The velocities are scaled with the linear wave group velocity, estimated from time series as \( C_{gr} = g/(2\omega_m) \). There are five markers for three
values of dimensionless upper amplitudes of the wave groups, \( A_c \omega_m^2/g \). By stars the scaled velocities of solitary wave groups observed in numerical simulations of hydrodynamic equations (see Sec. III) are plotted. Very good agreement between the laboratory and numerical results is found. Two pairs of the group amplitudes almost coincide in Fig. 9. These pairs correspond to the boundary conditions with the same steepness \( k_0 A_0 = 0.20 \) and \( k_0 A_0 = 0.30 \), respectively but with different phases of the wave group: “crest” or “trough” (see Table II). Hence, the resulting stable wave group is proved to be independent of the phase of the wavemaker signal; simultaneously, the laboratory measurements are shown to be repeatable.

The propagation speed of the NLS soliton (7) and (8) is equal to the linear wave group velocity and does not depend on the wave amplitude. Velocities of stationary wave groups observed in the laboratory and numerical experiments obviously depend on the amplitude as shown in Fig. 9. The velocity grows with amplitude; it exceeds about 11% the value of the group velocity of linear waves \( C_{gr} \) in the steepest laboratory case, and about 16% in the numerical simulations of the most extreme solitary group.
The dashed line in Fig. 9 shows the nonlinear velocity of Stokes waves (which is found numerically), normalized by the linear speed \( C_{gr} \). It is clear that the effect of nonlinearity on the speed of solitary waves is even more significant than the nonlinear correction to the Stokes wave velocity.

The general conclusion which may be made on the basis of all 43 experimental runs is that it is not obvious to establish which approach between Method 1 and Method 2 is the best. Both of these methods provide significantly better initial conditions than Method 3. As waves with \( k_0A_0 = 0.35 \) break in numerical simulations, for this steepness only Method 1 is able to provide an initial condition. We emphasize that Method 1 is the basic analytic formula for the exact envelope soliton solution of the NLS equation, when bound waves are disregarded. No differences between results from experiments characterized by boundary condition of a group with high crest or with a deep trough have been found.

The particular choice of the carrier frequency seems to be important for the successful generation of a steep solitary group; the reason for that is, however, not fully understood. The experimental conditions correspond to not very deep water; for longer carrier wave the effect of finite depth may become important (this seems to be the only substantial difference between experiment No. 30.29 and No. 30.37 in Table II). Simultaneously, variation of the mean wave frequency changes the relative limits of the frequency domain, which can be reproduced by the wave generator. Hence, this circumstance may also have effect on the wave generation.

![Graph](image-url)

**FIG. 9.** Velocities of stationary wave groups, observed in numerical simulations (stars), and of the selected “best” wave groups measured in laboratory experiments (circles). The speed of the uniform Stokes wave is given by the dashed line for the reference.
V. CONCLUSIONS

The existence of structurally stable short wave packets of steep waves is proved in laboratory experiments; they propagate without noticeable change of envelope shape for more than 60 wavelengths. To the best of our knowledge, this is the first time that such steep solitary wave groups (up to $A_{cr} \omega^2_m/l_g \approx 0.30$) have been observed in wave tank experiments. Our results are consistent with recent numerical findings where similar solitary groups were found in numerical simulations of the primitive Euler equations.13, 14 The laboratory measurements are compared with the results of strongly nonlinear numerical simulations of the Euler equations, where stationary wave groups are obtained as a result of long-time evolution of NLS envelope solitons.

Such short wave groups are the result of the balance between the nonlinearity and the dispersion of water waves. The difference between linear dispersive groups and the nonlinear solitary groups is hidden in the wave phases (relations between the elevation and the fluid velocities); such correlation is extremely hard to measure. We did our best to quantify the most tangible wave features with the purpose of comparing a Stokes wave (maybe, weakly modulated) and a wave in a solitary group: we have estimated the wave velocity and wave asymmetry.

Envelope curves estimated from the numerically obtained stationary wave groups fit the groups observed in the laboratory experiments very well. Velocities of the stable wave groups, obtained on the basis of laboratory and numerical experiments also agree very well. The numerical simulations reveal some spectral variability of wave groups, most likely due to the overlap between free and bound wave harmonics. Laboratory measurements show some stronger spectral variation for the most intense wave groups; its genesis is not clear.

When compared with uniform strongly nonlinear Stokes waves with the same frequency and crest amplitude, the observed solitary groups do not show distinctive difference in vertical asymmetry. However, the solitary groups move significantly faster than the Stokes waves; the effect of nonlinearity on their speed is even stronger than the nonlinear correction to the Stokes wave velocity.

Generally speaking, the NLS analytical soliton solution is found to be quite efficient for the generation of fairly steep solitary wave groups in laboratory. The observed groups correspond to the limiting case (very short and very intense) of envelope solitons. Results of laboratory experiments turn out to be sensitive with respect to the selection of the carrier wave frequency. This effect may happen due to wavemaker effects (different operating frequency), or caused by finite depth effects, i.e., by changing the carrier wave frequency, the dimensionless water depth, $k_0 h$, changes.

The wave breaking limits stationary wave groups in amplitude. Even steeper and shorter long-living wave packets are observed in the 2D numerical simulations (up to $A_{cr} \omega^2_m/l_g \approx 0.40$) in comparison with the laboratory observations ($A_{cr} \omega^2_m/l_g \approx 0.30$). These maximal groups are formed from NLS envelope solitons with steepness $A_0 \omega_0^2/l_g = 0.32$ and $A_0 \omega_0^2/l_g = 0.35$, respectively.

In laboratory experiments the accuracy of the initial (boundary) condition seems to be worse, although no obvious wave radiation, which could result in a wave group damping, was observed in the records of the “best” stable wave groups. For steeper waves the control signal for the wave maker via linear Response Amplitude Operators may become less accurate in comparison with less steep waves. Another important reason for the loss of accuracy of generation of steep solitary groups could be the geometry and type of wave board, i.e., depending on the wave board mode and geometry the velocity profile at the wave board differs significantly from the “natural” one under the same surface elevation. It should be stressed however, that other physical mechanisms could limit the maximum height of the quasi-unidirectional stationary group observed in the laboratory tank, such as 3D instability effects. Indeed, intense waves are known to suffer from 3D instabilities; however, the essentially strong modulation is the peculiarity of the considered waves and hence the direct adoption of results of stability analysis for uniform or weakly modulated waves is not possible. Conditions which led to wave breaking were not considered in the present work.

Simulations by Dyachenko and Zakharov13 did demonstrate the existence of similar extremely steep solitary wave groups numerically. In Ref. 13 two important simplifications were used: (i) purely 2D wave dynamics and (ii) potential form of the hydrodynamic equations. The present study confirms experimentally the existence of such steep waves, although not as steep as considered
in Ref. 13 in numerical experiments. If the reason for that is purely technical (such as imperfectness of the wave generator, etc.) or physical (for example, 3D instabilities), is not known by the moment.

The second significant result of our work is that such short steep wave groups (so short, that the concept of wave envelope cannot be straightforwardly introduced), may be generated in a wave tank with use of the classic NLS soliton solution.

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